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CSD-TR-973

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# **GOOD TRIANGULATIONS IN PLANE\***

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# Good Triangulations in Plane\*

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## 1 Introduction

Triangulation of a point set or a region in the plane is a very important problem since it has a number of applications in several areas. Though a lot of literature is available today on the topic, very few of them are addressed to the problem of guaranteed quality triangulation where the triangles are guaranteed to have some desirable qualities such as good bounds on their maximum and minimum angles. In fact, in finite element method, it is often desirable that the triangles have no obtuse angles. See [2]. Such triangles are called nonobtuse triangles. Any triangle which has an obtuse angle is called an obtuse triangle. Any triangulation which have only nonobtuse triangles is called a nonobtuse triangulation. Several triangulation methods, known today uses delaunay triangulation to start with and improve the triangulation with iterative refinements. It is known that if a given set of points admits a nonobtuse triangulation, the delaunay triangulation of the point set is nonobtuse. See [5].

In finite element method, the error bound is kept low if the triangles are as close as equilateral triangles. In [1], Babuska et. al showed that in finite element approximations with triangular elements, the smaller the maximum angle is, the lower the error bound becomes. Small angles are also not desirable since they give rise to ill-conditioned matrices [6]. Throughout this paper, by the boundary of a given set  $S$  of points, we mean (i) the boundary of the polygon  $P$  if  $S$  is given inside or on the boundary of  $P$  (ii) the boundary of the convex hull of  $S$ , otherwise. In [2], Baker et. al have given a method for nonobtuse triangulation of a polygonal region using new points inside and on the boundary of the polygon. Their algorithm provides a tedious method for triangulating the regions near boundary and do not work if the triangulation needs to include some prespecified input points inside the polygon, a requirement that often arises in interpolation techniques used for geological data. In [3] Chew has given an algorithm based on delaunay triangulation which triangulates a planar region and produces triangles with some guaranteed qualities. In this paper, we give two algorithms for good triangulations in plane. In the first algorithm, we follow the approach of Chew[3] which adds new points until a good delaunay triangulation is obtained. The choice of new points is important to produce guaranteed quality triangles without violating the constraint that no point is introduced outside the boundary. In the second algorithm, we follow the approach of Baker et. al[2] which fits an orthogonal grid over the given set of points and gives a method of triangulating each cell thus produced.

**Results:** We give an algorithm for triangulating a planar point set which may or may not be given inside a polygonal boundary. As stated earlier, if the point set is not given inside a polygon, we consider the convex hull of the given point set as the polygon containing it. Our first algorithm given in Section 3 produces triangles with the following ensured qualities.

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1. All obtuse triangles have angles between  $30^\circ$  and  $120^\circ$  and have sides of length in between  $d$  and  $2d$ .
2. All triangles with a boundary edge as one of the sides, have angles in between  $38.9^\circ$  and  $97.2^\circ$  and have sides of length in between  $d$  and  $1.5d$ .

The second algorithm given in Section 4 produces a triangulation in which all obtuse triangles have angles in between  $12^\circ$  and  $101^\circ$ . This algorithm is much more simpler than the one given in [2] and achieves reasonably good bounds on the angles. Moreover, it works not only for triangulating a polygon but also a point set given inside a polygon.

## 2 Voronoi Diagrams

### 2.1 Properties

Voronoi diagrams are well known data structures in computational geometry. Given a set  $S$  of points in the plane, voronoi diagram is the tessellation of the plane into regions, each containing one point such that every point in that region is closer to that point than any other given point. Two regions meet along an edge and two edges meet at a vertex. The edges and vertices of the voronoi diagram forms a plane graph. Its straight line dual gives a triangulation, called the delaunay triangulation of the given set of points. See Figure 2.1.

In what follows, the given points for voronoi tessellation are called generators or simply points and the vertices in the voronoi diagrams are called voronoi vertices. In non-degenerate situation, each voronoi vertex is adjacent to three regions containing three points. For each voronoi vertex, we call these three points as the corresponding points or generators for that voronoi vertex. At each voronoi vertex, the angles between adjacent voronoi edges are called the voronoi angles. Each generator corresponding to a voronoi vertex lies in one of the voronoi angles subtended at that voronoi vertex. See Figure 2.1.

Voronoi diagrams and delaunay triangulations admit some nice properties.

**Property 1:** Each region in a voronoi diagram is a convex region and two adjacent regions share only one voronoi edge

**Property 2:** The circumcircle of any triangle in a delaunay triangulation does not contain any other point inside. This is called the empty circle property.

**Property 3:** Let  $\overline{p_i p_j}$  be an edge in a triangulation  $T$  of a set of points. Let  $p_k$  and  $p_l$  be other two points of the triangles incident on  $\overline{p_i p_j}$ .  $T$  is a delaunay triangulation if and only if  $\angle p_i p_k p_j + \angle p_j p_l p_i < 180^\circ$  for every such inner edge of  $T$ .

**Property 4:** Among all triangulations of a set of points, delaunay triangulation minimizes the maximum angle.

In many cases, one needs a triangulation of a set of points which is as close as possible to a delaunay triangulation and which includes some prespecified edges. This is called Constrained Delaunay Triangulation(*CDT*). See [7]. Property 2 for *CDT* is modified as follows.

**Property 2(*CDT*):** If the circumcircle of any triangle in *CDT* contains a point inside, that point must be invisible from at least one of the three points forming the triangle. In other words, at least one of the line segments joining that point to one of the three points forming the triangle must intersect a prespecified edge.

Property 4 is valid for *CDT* also, though property 3 is not.

### 2.2 Definitions

For any triangulation of a set of points, we call those edges as the boundary edges which have only one triangle incident on them. For a given set  $S$  of points in a polygonal region, we consider Constrained

Delaunay Triangulation which includes the edges of the given polygon. These edges appear as boundary edges of the triangulation. If the set  $S$  is not given inside a polygon, the boundary edges of the delaunay triangulation of  $S$  are the edges of the convex hull of the given point set.

A triangle in any triangulation is called a boundary triangle if it has a boundary edge as its side. The triangles which are not boundary triangles are called inner triangles.

A triangle in any triangulation  $T$  is said to have good circumcenter if and only if the line segments joining the circumcenter and three vertices of the triangle do not intersect any boundary edge of  $T$ . Conversely, a triangle is said to have bad circumcenter if and only if one of the line segments joining the circumcenter to its vertices intersect a boundary edge of  $T$ .

## 2.3 Geometric Lemmas

The following lemmas are used in the next section.

**Lemma 2.1:** Let  $\overline{p_i p_j}$  and  $\overline{p_x p_y}$  be two chords drawn in a semicircle  $U$ . Further, let  $\overline{p_x p_y}$  lie in between  $\overline{p_i p_j}$  and the diameter of  $U$ . Then,  $|\overline{p_x p_y}| \geq |\overline{p_i p_j}|$

**Proof:** Obvious. ♣

**Lemma 2.2:** Let  $\overline{p_i p_j}$  be a chord in a circle  $C$  and  $p_m$  be a point which lies outside  $C$  and on the same side of  $\overline{p_i p_j}$  as the center of  $C$  does. Further, let  $\angle p_m p_i p_j$  and  $\angle p_i p_j p_m$  be nonobtuse. Then center of  $C$  lies inside  $\triangle p_i p_j p_m$ .

**Proof:** Let  $p_i \tilde{p}_i$  and  $p_j \tilde{p}_j$  be two perpendicular rays drawn on  $\overline{p_i p_j}$  at  $p_i$  and  $p_j$  respectively extending to the same side of  $\overline{p_i p_j}$  which contains the center of  $C$ . See Figure 2.2. Let  $p_i \tilde{p}_i$  and  $p_j \tilde{p}_j$  intersect  $C$  at  $p_t$  and  $p_s$ . Certainly,  $\overline{p_i p_t}$  and  $\overline{p_j p_s}$  are two diameters of  $C$ .  $p_m$  can only lie in the shaded region as shown in Figure 2.2. It is obvious that, for any point  $p_m$  in the shaded region, the triangle  $\triangle p_i p_j p_m$  contains the center of  $C$ . ♣

**Lemma 2.3:** Let  $G = \triangle p_i p_j p_k$  be an obtuse triangle with bad circumcenter in any delaunay triangulation. Let  $\overline{p_i p_k}$  be the largest side of  $\triangle p_i p_j p_k$ . There must exist a boundary edge which is greater than or equal to  $\overline{p_i p_k}$ .

**Proof:** Let  $C$  be the circumcircle of  $G$  with the circumcenter  $v_l$ . Since  $G$  has bad circumcenter, there must be a boundary edge with endpoints outside or on  $C$  and intersecting  $\overline{p_i v_l}$  and  $\overline{p_k v_l}$  as shown in Figure 2.3. Let  $\overline{p_u p_v}$  be such a boundary edge which intersects  $C$  at  $p_x$  and  $p_y$  and is closest to  $\overline{p_i p_k}$ . Draw the diameter  $\overline{p_t p_s}$  of  $C$  which is parallel to  $\overline{p_u p_v}$ . Applying Lemma 2.1 to the chords  $\overline{p_i p_k}$  and  $\overline{p_x p_y}$  we get

$$|\overline{p_u p_v}| \geq |\overline{p_x p_y}| \geq |\overline{p_i p_k}|.$$

♣

## 3 Using Delaunay Triangulation

### 3.1 Algorithms

Let  $S$  be a set of points given in the plane. Let  $G = \triangle p_i p_j p_k$  be a triangle in the delaunay triangulation of  $S$ .  $G$  is obtuse if and only if its circumcenter lies outside. we may have a very simple algorithm *TRI1* for nonobtuse triangulation of a planar point set as follows.

*Algorithm TRI1:*

*begin*

*Construct the delaunay triangulation of the given points*

*Repeat*

*Add the circumcenter of an obtuse triangle.*

*update the current triangulation.*

*Until there is no such triangle.*

*end.*

The above algorithm certainly gives a nonobtuse triangulation if it terminates. But, unfortunately, termination is not guaranteed and moreover it may add points outside the boundary of the given set of points which may not be desirable.

We modify the algorithm *TRI1* to have another algorithm *TRI2* where we take care of these two problems though we can not guarantee a nonobtuse triangulation. However, we ensure that all the obtuse triangles produced by our algorithm have some guaranteed qualities. Two input conditions must be satisfied for the algorithm *TRI2*. Later, we will see how these conditions are met.

*Algorithm TRI2:*

Input Conditions: Let  $d$  be a quantity, such that no two given points are closer than  $d$  and no boundary edge is greater than  $1.5d$  and less than  $d$ .

*begin*

*Construct the delaunay triangulation of the given points.*

*Repeat*

*Add the circumcenter  $v_l$  corresponding of  $G = \Delta p_i p_j p_k$  satisfying following properties.*

*1.  $G$  is obtuse.*

*2.  $v_l$  is at a distance of at least  $d$  from all the three points  $p_i, p_j, p_k$ .*

*3.  $v_l$  is a good circumcenter of  $G$ .*

*Update the current triangulation.*

*Until there is no such triangle.*

*end*

### 3.2 Analysis of Algorithm TRI2

**Lemma 3.1:** Algorithm *TRI2* terminates.

**Proof:** By the empty circle property of the delaunay triangulation, the added points are at a distance of at least  $d$  from all other points. Since, there can be finitely many such points in a bounded region, algorithm *TRI1* can add finite number of new points and thus terminates. ♣.

**Lemma 3.2:** Each obtuse triangle  $G$  with good circumcenter produced by *TRI2* have the following criteria.  $G$  has no angles greater than  $120^\circ$  and less than  $30^\circ$  and moreover,  $G$  has no edge greater than  $2d$  and less than  $d$ .

**proof:** Let  $G = \Delta p_i p_j p_k$  be an obtuse triangle with good circumcenter  $v_l$ . Let  $\angle p_i p_j p_k$  be the obtuse angle. Since the good circumcenter  $v_l$  has not been introduced by *TRI2*, its distance from  $p_i, p_j, p_k$  must be less than  $d$ . See Figure 3.1. Let  $\angle p_j p_k p_i$  be the minimum angle in  $\Delta p_i p_j p_k$ . Consider the triangle  $\Delta p_i p_j v_l$ . Since  $\angle p_j v_l p_i = 2\angle p_j p_k p_i$ , we have,  $(\angle p_j p_k p_i)_{\min} = \frac{1}{2}(\angle p_j v_l p_i)_{\min}$ . In  $\Delta p_i p_j v_l$ ,  $|\overline{p_i p_j}| > d$  and  $|\overline{p_i v_l}| = |\overline{p_j v_l}| < d$ . Thus, we have

$$2\angle p_i p_j v_l + \angle p_j v_l p_i = 180^\circ$$

$$\begin{aligned} 2\angle p_i p_j v_l + \angle p_i p_j v_l &< 180^\circ \\ \angle p_i p_j v_l &< 60^\circ \end{aligned}$$

This gives ,

$$(\angle p_j p_k p_i)_{\min} = 30^\circ$$

Minimum angle of  $30^\circ$  assures a maximum angle of  $120^\circ$  in the triangle  $G$ . Since the circumcenter of the triangle  $G$  has a radius less than  $d$ , each side of  $G$  must be less than  $2d$ . This with the fact that no two points in the triangulation produced by  $TRI2$  are closer than  $d$  ensures that each side of  $G$  is in between  $d$  and  $2d$ . ♣

**Lemma 3.3:** Let  $G = \triangle p_i p_j p_k$  be a triangle produced by  $TRI2$  which satisfies the following conditions.

1.  $G$  is obtuse with the obtuse angle  $\angle p_i p_j p_k$ .
2.  $G$  has good circumcenter.
3.  $|\overline{p_i p_j}| \geq \sqrt{2}d$  and  $|\overline{p_j p_k}| \geq d$ .

Let  $\overline{p_j p_l}$  intersect  $\overline{p_i p_k}$  inside at  $p_l$  and  $\angle p_i p_j p_l$  be obtuse as shown in Figure 3.2. Then,  $|\overline{p_i p_l}| > 1.58d$

**Proof:** Since  $G$  is an obtuse triangle with a good circumcenter, the minimum angle in  $G$  is greater than  $30^\circ$  by Lemma 3.2. Drop a perpendicular  $\overline{p_j p_n}$  from  $p_j$  on  $\overline{p_i p_k}$  as shown in Figure 3.2. From triangle  $\triangle p_i p_j p_n$  we get,

$$\begin{aligned} |\overline{p_i p_j}| &= \frac{|\overline{p_j p_n}|}{\sin \angle p_n p_i p_j} \\ |\overline{p_j p_n}| &\geq (|\overline{p_i p_j}| \sin \angle p_n p_i p_j)_{\min} \\ &\geq \sqrt{2}d \sin 30^\circ \\ &= \frac{d}{\sqrt{2}} \end{aligned}$$

Since  $|\overline{p_j p_l}| > |\overline{p_j p_n}|$ , we have,

$$|\overline{p_j p_l}| > \frac{d}{\sqrt{2}}$$

Since  $\angle p_i p_j p_l$  is obtuse, we have

$$\begin{aligned} |\overline{p_i p_l}| &> \sqrt{|\overline{p_i p_j}|^2 + |\overline{p_j p_l}|^2} \\ &> \sqrt{\frac{5}{2}}d \\ &> 1.58d \end{aligned}$$

♣

**Lemma 3.4:** Any triangle  $G$  produced by  $TRI2$  which has bad circumcenter is a boundary triangle. Moreover,  $G$  has a boundary edge as the opposite side of the obtuse angle.

**Proof:** Let  $G = \triangle p_i p_j p_k$  with the circumcircle  $C$  have the bad circumcenter. Assume  $\triangle p_i p_j p_k$  to be an inner triangle. We show that this leads to a contradiction. Since  $G$  has a bad circumcenter, the line segment joining one of the vertices of  $G$  and its circumcenter must intersect a boundary edge. Let  $\overline{p_u p_v}$  be that boundary edge. Obviously, circumcenter of  $G$  can not lie inside it. Hence,  $G$  is an obtuse triangle. Let  $\angle p_i p_j p_k$  be the obtuse angle as shown in Figure 3.3. Since each side of the triangle is greater than or



equal to  $d$ , the side  $\overline{p_i p_k}$  is greater than or equal to  $\sqrt{2}d$ . Further, the side  $\overline{p_i p_k}$  can not be a boundary edge since  $\Delta p_i p_j p_k$  is an inner triangle. Hence,  $\overline{p_i p_k}$  is adjacent to another triangle, say  $\Delta p_i p_k p_m$ . We claim that one of the angles  $\angle p_i p_k p_m$ ,  $\angle p_m p_i p_k$  is obtuse. Suppose they are both nonobtuse. This with the condition that  $p_m$  lies outside  $C$  asserts that the center of  $C$  lies inside  $\Delta p_i p_k p_m$  by Lemma 2.2. Then one of the sides  $\overline{p_i p_m}$ ,  $\overline{p_k p_m}$  intersects the boundary edge  $\overline{p_u p_v}$ , which is impossible. Hence, one of the angles  $\angle p_i p_k p_m$ ,  $\angle p_m p_i p_k$  must be obtuse. W.l.o.g., assume  $\angle p_i p_k p_m$  is obtuse. In the obtuse triangle  $\Delta p_i p_k p_m$ ,  $|\overline{p_i p_k}| \geq \sqrt{2}d$  and  $|\overline{p_k p_m}| \geq d$ . This implies  $|\overline{p_i p_m}| \geq 3d$ . Now, consider the following two cases.  
*Case(i):*  $\Delta p_i p_k p_m$  has a good circumcenter. Let  $\overline{p_i p_m}$  intersect  $C$  at  $p_l$ . Since  $\overline{p_i p_l}$  must lie on that side of the boundary edge  $\overline{p_u p_v}$  which does not contain the center of  $C$ , the triangle  $\Delta p_i p_k p_l$  must be obtuse with  $\angle p_i p_k p_l > 90^\circ$ . Thus,  $\Delta p_i p_k p_m$  and  $\overline{p_k p_l}$  satisfy all the conditions stated in Lemma 3.3. Hence,  $|\overline{p_i p_l}| > 1.58d$ . Thus, we have

$$|\overline{p_u p_v}| > |\overline{p_i p_l}| > 1.58d$$

But, according to the input conditions of *TRI2*,  $\overline{p_u p_v}$  must be less than  $1.5d$ , a contradiction.

*Case(ii):*  $\Delta p_i p_k p_m$  does not have good circumcenter. The largest side  $\overline{p_i p_m}$  of  $\Delta p_i p_k p_m$  is greater than  $1.5d$ . By Lemma 2.3, there must be a boundary edge of length greater than  $1.5d$ , an impossibility.

Hence, our assumption that  $\Delta p_i p_j p_k$  is an inner triangle must be wrong. Assume  $\Delta p_i p_j p_k$  be a boundary triangle with the side  $\overline{p_i p_k}$  not being a boundary edge. We can argue in the same way as above to reach a contradiction. Hence,  $\overline{p_i p_k}$  must be a boundary edge. Therefore,  $G = \Delta p_i p_j p_k$  must be a boundary triangle with the side opposite to the obtuse angle as a boundary edge. ♣

**Lemma 3.5:** Each inner obtuse triangle  $G$  produced by *TRI2* satisfies the following conditions.

1. Each angle of  $G$  is in between  $30^\circ$  and  $120^\circ$ .
2. Each side of  $G$  has length in between  $d$  and  $2d$ .

**Proof:** Combine Lemma 3.2 and Lemma 3.4. ♣

**Lemma 3.6:** Let  $G = \Delta p_i p_j p_k$  be a triangle such that each of its sides has length between  $d$  and  $1.5d$ . Then, the angles of  $\Delta p_i p_j p_k$  must be in between  $38.9^\circ$  and  $97.2^\circ$ .

**Proof:** Let  $\theta(G)$  and  $\Theta(G)$  denote the minimum and maximum angle of the triangle  $G$ . We claim that any triangle  $G$  having minimum  $\theta(G)$  among all triangles satisfying the stated conditions on side lengths must have a side equal to  $1.5d$ . Suppose, on the contrary,  $G$  does not have a side equal to  $1.5d$ . Let  $\overline{p_i p_k}$  be the largest side in  $G$ . Stretching  $\overline{p_i p_k}$  up to the length of  $1.5d$  decreases  $\theta(G)$ , which contradicts our assumption. Thus,  $G$  must have one side equal to  $1.5d$ . Similarly, we can prove that any  $G$  having maximum  $\Theta(G)$  among all triangles satisfying the constraints on side lengths must have a side equal to  $1.5d$ . Now, we prove that any triangle  $G$  having minimum  $\theta(G)$  and satisfying the stated conditions on side lengths must have two sides of length  $1.5d$  and another side of length  $d$ . As we proved  $G$  must have a side of length  $1.5d$ . Let  $\overline{p_i p_k}$  be that side.  $\overline{p_i p_k}$  must be larger than or equal to other two sides. Let the two circles drawn with radius  $1.5d$  and centers at  $p_i, p_k$  meet at  $p_l, p_m$  respectively as shown in the Figure 3.4. Let other two circles drawn with radius  $h$  and with centers at  $p_i, p_k$  meet at  $p_n, p_q$  respectively as shown in the figure 3.4. To satisfy the constraints on the lengths of the sides of  $G = \Delta p_i p_j p_k$ ,  $p_j$  must lie in the shaded region. It is easy to see that if  $p_j$  is placed at any one position of  $p_a, p_b, p_c, p_d$ ,  $\theta(\Delta p_i p_j p_k)$  gets minimized. With this position of  $p_j$ ,  $G$  has two sides equal to  $1.5d$  and another side equal to  $d$ . It is easy to see that the minimum angle in such a triangle is greater than  $38.9^\circ$ . As one can observe, the maximum angle in  $G$  is maximized when  $p_j$  is placed at any position of  $p_n, p_q$ . The maximum angle in such a triangle is less than  $97.2^\circ$ . ♣

**Theorem 3.1:** Triangles produced by the algorithm *TRI2* satisfies the following conditions.

1. Each inner obtuse triangle has angles in between  $30^\circ$  and  $120^\circ$  and sides of length in between  $d$  and  $2d$ .

2. Each boundary obtuse triangle has angles in between  $38.9^\circ$  and  $97.2^\circ$  and sides of length in between  $d$  and  $1.5d$ .

**Proof:** According to Lemma 3.4 each obtuse boundary triangle has a boundary edge as the opposite side of the obtuse angle. Since each boundary edge has length in between  $d$  and  $1.5d$ , these triangles must have sides of length in between  $d$  and  $1.5d$ . Further, according to Lemma 3.6 these triangles have angles in between  $38.9^\circ$  and  $97.2^\circ$ . By Lemma 3.5, each inner obtuse triangle produced by *TRI2* satisfies the stated conditions. ♣

**Input Conditions of TRI2:** Let  $\delta_1$  be the minimum distance between any two points. Let  $\delta_2$  be the minimum distance between a point and a boundary edge and  $\delta_3$  be the minimum length of a boundary edge. Let  $d = \min(\delta_1, \delta_2, \frac{\delta_3}{3})$ . Definitely, each boundary edge is greater than or equal to  $3d$ . It is easy to divide such edges into segments which have lengths in between  $d$  and  $1.5d$ . This introduces new points which can not be closer than  $d$  to any other points by the choice of  $d$ . Again, no two points can be closer than  $d$ . Thus,  $d$  satisfies all the input conditions of the algorithm *TRI2*.

**Complexity of TRI2:** The time complexity of the algorithm depends on the time of updation. Clearly, each updation can be done in  $O(n)$  time where  $n$  is the number of points present in the output. This gives a time complexity of  $O(n^2)$ , though on the average, the number of points affected by each update remains constant and thus the algorithm runs in  $O(n)$  time on the average.

Let the area of the boundary to be triangulated be  $A$ . Since the distance between any two points is at least  $d$  in the output produced by the algorithm, the number of points added is bounded by the number of equilateral triangles with sides of length  $d$  which can fit in the area  $A$ . Thus, *TRI2* produces at most  $\frac{4A}{\sqrt{3}d^2}$  triangles.

## 4 Using Grid

In [2], Baker et. al have given an algorithm to triangulate a simple polygon with nonobtuse triangles. They overlay a square grid on the polygon and observe that each inner square through which no boundary edge passes, can be triangulated with two right angled triangles. The difficult part is to triangulate the squares through which a boundary edge passes. See Figure 4.1. They give a tedious method to triangulate these regions into nonobtuse triangles. Here, we give a very simple method to triangulate these regions so that all obtuse triangles have angles between  $12^\circ$  and  $101^\circ$ . Our algorithm allows input points inside and on the boundary of the given polygon.

### 4.1 Geometric Lemmas

**Lemma 4.1:** Let  $\overline{p_i p_j}$  be a line segment. Let  $p_i \vec{p_s}$  and  $p_j \vec{p_t}$  be two rays, perpendicular to  $\overline{p_i p_j}$  at  $p_i$  and  $p_j$ . Further, let  $\overline{p_l p_m}$  be another line segment, parallel to  $\overline{p_i p_j}$  intersecting  $p_i \vec{p_s}$  and  $p_j \vec{p_t}$  at  $p_l$  and  $p_m$  respectively. Let  $p_k$  be a point which lie in the shaded region as shown in Figure 4.2. The angle  $\angle p_j p_k p_i$  is maximized when  $p_k$  is the midpoint of the line segment  $\overline{p_l p_m}$ .

**Proof:** We first prove that when  $\angle p_j p_k p_i$  is maximized,  $p_k$  lies on  $\overline{p_l p_m}$ . Suppose, it does not. Drop a perpendicular from  $p_k$  on  $\overline{p_l p_m}$  which intersects  $\overline{p_l p_m}$  at  $p'_k$ . Certainly,  $\angle p_j p'_k p_i > \angle p_j p_k p_i$ . Hence,  $\angle p_j p_k p_i$  can not be maximized when  $p_k$  does not lie on  $\overline{p_l p_m}$ . Now, we prove that among all positions of  $p_k$  on  $\overline{p_l p_m}$ , the midpoint of  $\overline{p_l p_m}$  maximizes  $\angle p_j p_k p_i$ . Let  $|\overline{p_i p_l}| = S$ ,  $|\overline{p_l p_k}| = x$  and  $|\overline{p_i p_j}| = T$ . We have,

$$\angle p_j p_k p_i = 180^\circ - \tan^{-1}\left(\frac{S}{x}\right) - \tan^{-1}\left(\frac{S}{T-x}\right)$$

$\angle p_j p_k p_i$  is maximized when  $\tan^{-1}\left(\frac{S}{x}\right) + \tan^{-1}\left(\frac{S}{T-x}\right)$  is minimized at  $x = \frac{T}{2}$ , which proves the lemma.

**Lemma 4.2:** Let  $\overline{p_i p_j}$  be a line segment.  $p_i \tilde{p}_s$  and  $p_j \tilde{p}_t$  are two rays perpendicular to  $\overline{p_i p_j}$  drawn at  $p_i$  and  $p_j$  respectively. Let  $p_l, p_m$  and  $p_k$  be three points on  $p_i \tilde{p}_s, p_j \tilde{p}_t$  and  $\overline{p_i p_j}$  respectively as shown in Figure 4.3. Further, let  $|\overline{p_i p_l}| = S$ ,  $|\overline{p_j p_m}| = L$  and  $|\overline{p_i p_j}| = T$ . Let  $S_{min}, L_{min}$  denote minimum values of  $S$  and  $L$  respectively and  $T_{max}$  denote the maximum value of  $T$ . The maximum value of  $\angle p_m p_k p_l$  is

$$180 - \tan^{-1}\left(\frac{S_{min}}{x_0}\right) - \tan^{-1}\left(\frac{L_{min}}{T_{max} - x_0}\right)$$

where

$$x_0 = \frac{\sqrt{T_{max}^2 S_{min}^2 + (L_{min} - S_{min})(T_{max}^2 S_{min} + L_{min} S_{min}(L_{min} - S_{min}))} - T_{max} S_{min}}{(L_{min} - S_{min})}$$

**Proof:** We first prove that when  $\angle p_m p_k p_l$  is maximized then  $|\overline{p_i p_l}| = S_{min}$ ,  $|\overline{p_j p_m}| = L_{min}$  and  $|\overline{p_i p_j}| = T_{max}$ . For any given  $T$  and  $L$ , it is easy to see that when  $\angle p_m p_k p_l$  is maximum,  $|\overline{p_i p_l}|$  must be equal to  $S_{min}$ . Suppose, on the contrary, for a given  $T$  and  $L$ ,  $|\overline{p_i p_l}| > S_{min}$ , when  $\angle p_m p_k p_l$  is maximum. Then certainly, bringing the point  $p_l$  closer to  $p_i$ , increases the angle  $\angle p_m p_k p_l$ . Similarly, we can prove that for given  $S$  and  $T$ , when  $\angle p_m p_k p_l$  is maximum,  $|\overline{p_j p_m}| = L_{min}$ . Now, we prove that for given  $S$  and  $L$ ,  $|\overline{p_i p_j}| = T_{max}$ , when  $\angle p_m p_k p_l$  is maximum. Suppose, on the contrary,  $|\overline{p_i p_j}| < T_{max}$  when  $\angle p_m p_k p_l$  is maximum. It is obvious that, by stretching  $\overline{p_i p_j}$  one can increase the angle  $\angle p_m p_k p_l$  which leads to the contradiction. All these together imply,  $|\overline{p_i p_j}| = T_{max}$ ,  $|\overline{p_i p_l}| = S_{min}$  and  $|\overline{p_j p_m}| = L_{min}$  when  $\angle p_m p_k p_l$  is maximum. Given any  $L, S$  and  $T$ , it is easy to see that

$$\angle p_m p_k p_l = 180^\circ - \tan^{-1}\left(\frac{S}{x}\right) - \tan^{-1}\left(\frac{L}{T - x}\right)$$

where  $|\overline{p_i p_k}| = x$ . By simple calculus, it can be shown that for a given  $L, S$  and  $T$ ,  $\angle p_m p_k p_l$  reaches maximum when

$$x = \frac{\sqrt{T^2 S^2 + (L - S)(T^2 S + L S(L - S))} - T S}{(L - S)}$$

This with the fact that  $T = T_{max}, L = L_{min}$  and  $S = S_{min}$  when  $\angle p_m p_k p_l$  reaches maximum proves the lemma.

## 4.2 Algorithm and Lemmas

**Lemma 4.3:** Let  $S$  be a set of points in the simple polygon  $P$  containing no acute interior angles.  $S$  includes the points corresponding to the vertices of the polygon. By introducing points inside  $P$  and on the boundary of  $P$ ,  $S$  can be triangulated in such a way that each obtuse triangle has angles between  $12^\circ$  and  $101^\circ$ .

**Proof:** We draw horizontal and vertical lines through each point in  $S$ . This forms a rectangular grid. We refine the grid so that through any rectangle no two nonadjacent boundary edges pass. We refine the grid further so that each side of any rectangle has length in between  $d$  and  $1.5d$  for some  $d$ . The choice of  $d$  and the method of this refinement are discussed later. We introduce the gridpoints which are inside  $P$  and where two gridlines intersect and also the points where the gridlines intersect the boundary. We introduce the edges between these points which are on the grid. Each internal rectangle through which no edge passes, can be triangulated into two right angled triangles by a diagonal. While triangulating the rectangles through which a boundary edge passes, we introduce points only inside  $P$  or on the boundary of  $P$ , but not on the sides of the rectangles. Thus, each rectangle can be triangulated independently without propagating points to the adjacent rectangles.

If two boundary segments pass through a rectangle, they must be adjacent. Since the interior angles between the corresponding boundary edges is obtuse, the regions of  $P$  bounded by these two boundary segments and the sides of the rectangle must be disjoint. Thus, we can triangulate these two regions independently. This implies that we need to worry about how to triangulate the region in  $P$  bounded by a boundary segment and sides of the rectangle without introducing points on the sides of the rectangle except at the corner points and the points where boundary intersects them.

Let  $abcd$  be a rectangle through which the boundary segment  $\overline{pq}$  passes. W.l.o.g., assume  $b$  to lie inside  $P$ .

*Case(i):* See Figure 4.4. The triangle is a nonobtuse triangle.

*Case(ii):* See Figure 4.5. In this case carry out the triangulation as shown. The angle  $\angle qpb$  is nonobtuse since  $q$  lies outside the circle drawn with the diameter  $\overline{ab}$ . This is because of the fact that the maximum length of  $\overline{ab}$  is  $1.5d$  and the minimum length of  $\overline{cd}$  is  $d$ .

*Case(iii):* See Figure 4.6. Let  $|\overline{ab}| = T$  and  $|\overline{ad}| = L$ . Without loss of generality, assume  $T \geq L$ . Draw a line segment  $\overline{st}$  which is parallel to  $\overline{ab}$  and at a distance of  $\frac{T}{2}\tan 40^\circ$  from it. We have two subcases depending on the position of  $p$ . Consider the case where  $p$  lies on  $\overline{as}$ . If  $\angle qpb$  is obtuse, carry out the triangulation as shown in Figure 4.6(a). Let  $\overline{qu}$  be the perpendicular line to  $\overline{cd}$  at  $q$  and  $\overline{pu}$  be the perpendicular line to  $\overline{pb}$  at  $p$ . These two lines meet at  $u$ . Join  $u$  to  $b$  and  $c$ . This may render  $\angle pub$  to be obtuse which can be resolved by dropping a perpendicular from  $u$  on  $\overline{pq}$ . We prove that the angle  $\angle cub$  can never be obtuse. As can be seen

$$\angle qpb = 90^\circ + \beta - \alpha$$

For  $\angle qpb$  to be obtuse, we must have  $\beta > \alpha$  which implies

$$\begin{aligned} \frac{|\overline{pd}|}{|\overline{dq}|} &> \frac{|\overline{ab}|}{|\overline{ap}|} \\ |\overline{dq}| &< \frac{|\overline{ap}||\overline{pd}|}{|\overline{ab}|} \end{aligned}$$

Let  $|\overline{ap}| = x$ . We have  $|\overline{dq}| < \frac{x(L-x)}{T}$ . This gives  $|\overline{dq}| < \frac{L^2}{4T} \leq \frac{T^2}{4T} = \frac{T}{4}$ . Thus maximum value of  $|\overline{dq}|$  is  $0.375d$ . This immediately implies  $u$  lies outside the semicircle drawn with the diameter  $\overline{bc}$ . Hence,  $\angle cub$  must be nonobtuse.

Consider the case when instead of  $\angle qpb$ , the angle  $\angle bqp$  is obtuse. We know the minimum values of  $|\overline{bc}|$  and  $|\overline{pd}|$  are  $d$  and  $(d - \frac{1.5d \tan 40^\circ}{2})$  respectively and the maximum value of  $|\overline{dc}|$  is  $1.5d$ . Applying Lemma 4.2 with these values we get the maximum value of  $\angle bqp$  to be less than  $101^\circ$ . It can be proved that minimum values of  $\angle qbp$  and  $\angle qpb$  are greater than  $12^\circ$ .

Let us consider the subcase when  $p$  lies on  $\overline{ds}$ . If the angle  $\angle qpb$  is obtuse, one can obtain nonobtuse triangulation in the same way as discussed in the previous subcase. If instead,  $\angle bqp$  is obtuse, carry out the triangulation as in the case when  $\angle qpb$  is obtuse, but with the role of  $p$  and  $q$  switched. See Figure 4.6(c). In this case the only angle which may be obtuse is  $\angle bua$ . Since  $|\overline{as}| = \frac{|\overline{ab}|\tan 40^\circ}{2}$ , the angle  $\angle bua$  is  $100^\circ$  when  $u$  is the midpoint of  $\overline{st}$ . Thus, by Lemma 4.1  $\angle bua$  has the maximum value of  $100^\circ$ . It easy to see that the minimum value of angles  $\angle uab$  and  $\angle abu$  is greater than  $\tan^{-1}(\frac{\tan 40^\circ}{2})$  which is greater than  $22^\circ$ .

This exhausts all possible cases and we observe that all obtuse triangles produced by this method have angles between  $12^\circ$  and  $101^\circ$ . ♣

**Generating grids with proper spacings:** As pointed out earlier, we need a rectangular grid which fit on the points inside and on the vertices of the polygon in such a way that the distance between two adjacent grid lines is in the range of  $d$  and  $1.5d$  for some  $d$ . Draw a horizontal and a vertical line through

each point to have an initial grid and then refine this grid so that no two nonadjacent boundary segments pass through a rectangle. Let  $h$  be the minimum spacing between any two adjacent grid lines. Take  $d = \frac{h}{3}$ . Definitely, with this choice of  $d$ , two adjacent gridlines have spacing of at least  $3d$ . It is easy to refine such a grid so that every adjacent gridline spacing lies between  $d$  and  $1.5d$ .

**Theorem 4.1:** Let  $S$  be a set of points in the simple polygon  $P$ .  $S$  includes the points corresponding to the vertices of the polygon. By introducing points inside  $P$  and on the boundary of  $P$ ,  $S$  can be triangulated in such a way that each obtuse triangle has angles between  $101^\circ$  and  $12^\circ$ .

**Proof:** As discussed in [2], corresponding to each vertex of  $P$  where the interior angle is acute, we can cut off a triangular portion, such that the cut off triangle is nonobtuse and does not contain any point inside. After this modification, we apply Lemma 4.3 on the new polygon thus generated from  $P$  which does not have any acute interior triangle. This may introduce points on the side of the cut off triangles which is not a boundary segment. Such triangles with those added points can be triangulated with nonobtuse triangles as shown in Figure 4.7.

## 5 Conclusion

Currently, research is going on to find out algorithms for good triangulations in 3-D. The algorithm based on delaunay triangulation can easily be extended to 3-D which guarantees the face angles of each tetrahedron to be in between  $30^\circ$  and  $120^\circ$ . But, this does not ensure any good bounds on the dihedral angles or solid angles of the tetrahedra. Natural extension of the second algorithm in 3-D is to fit a three dimensional rectangular grid on the given point set and triangulate each inner rectangular cell into 5 nonobtuse tetrahedra. As in 2-D, the difficult part is to triangulate the rectangular cell through which a boundary face passes. Research is going on to carry out this triangulation to ensure good bounds on the dihedral and solid angles of the tetrahedra.

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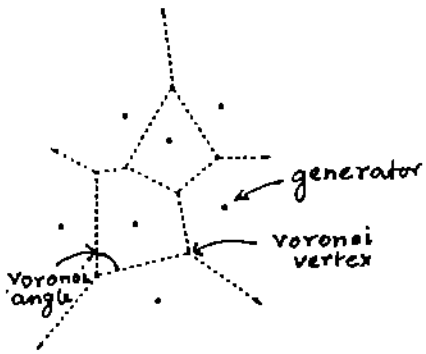


Figure 2.1

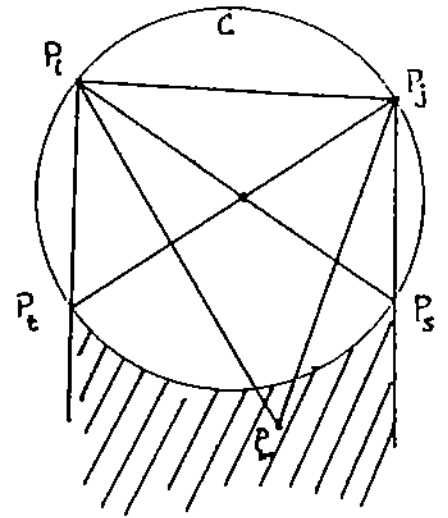


Figure 2.2

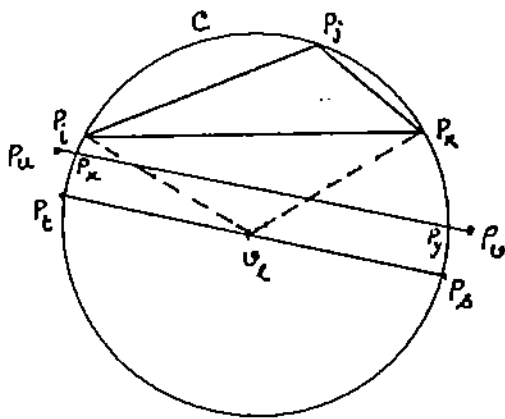


Figure 2.3

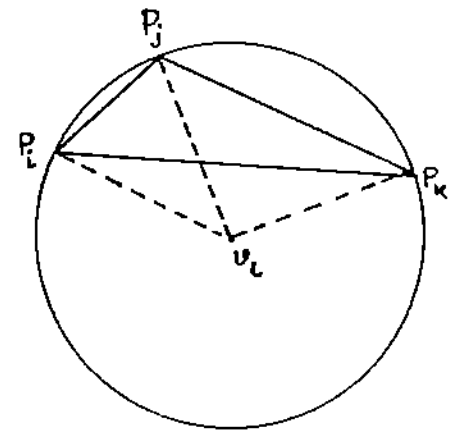


Figure 3.1

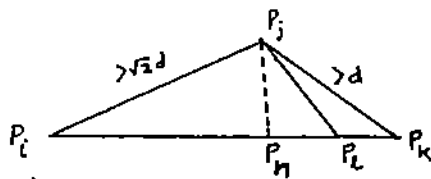


Figure 3.2

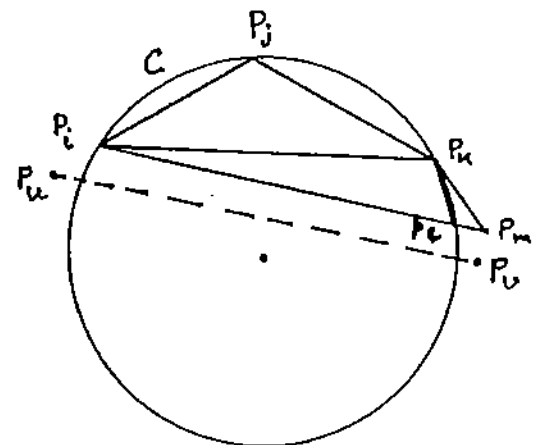


Figure 3.3

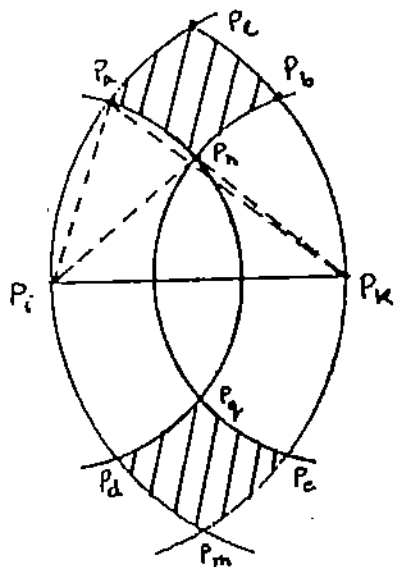


Figure 3.4

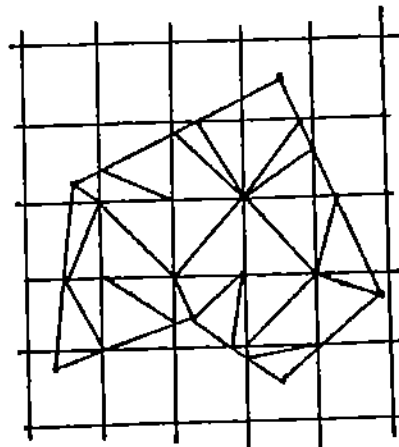


Figure 4.1

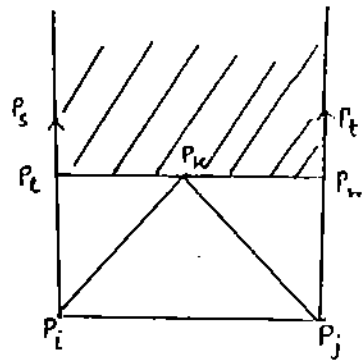


Figure 4.2

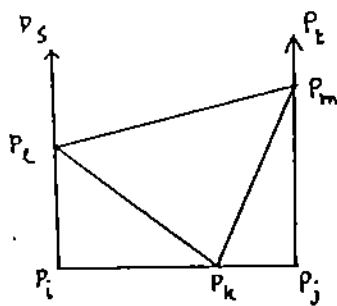


Figure 4.3

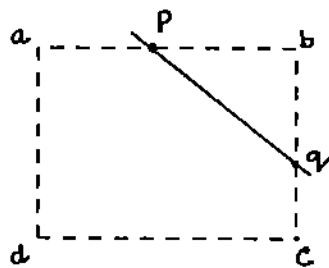


Figure 4.4

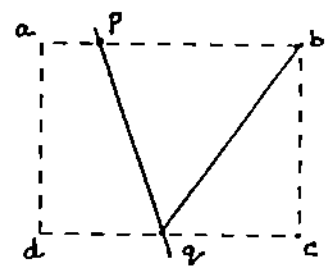


Figure 4.5

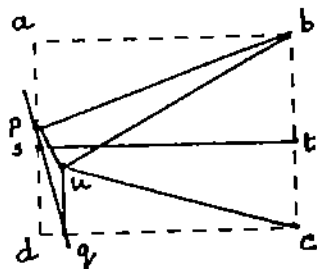


Figure 4.6(a)

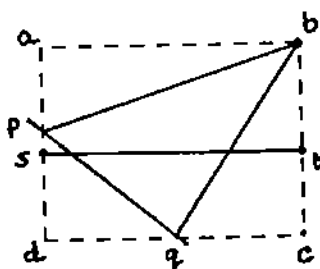


Figure 4.6(b)

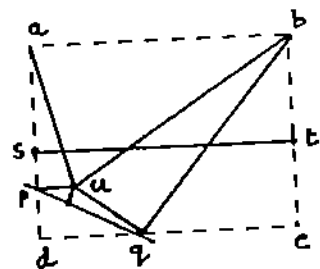


Figure 4.6(c)



Figure 4.7